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EQUILIBRIUM PROBLEM OF AN ELASTIC PLATE WITH AN OBLIQUE CRACK

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Equilibrium problems of elastic plates having vertical cracks (cuts) have been thus far studied in sufficient detail [1–3]. In the present paper, the conditions of mutual impenetrability of the sides of an oblique crack in the Kirchhoff–Love plate are obtained, equilibrium problems of a plate are formulated, and the main difficulties that arise in studying similar problems are discussed. As it turns out, the impenetrability condition for an oblique crack is of a nonlocal character in the sense that its expression for a given fixed point contains the values of plate displacements both at this point and at a point on the opposite side of the crack. This circumstance makes the impenetrability condition obtained substantially different from the corresponding condition for vertical cracks. In particular, unlike vertical cracks where the equilibrium conditions are satisfied at all points of the middle surface, the equilibrium conditions here are satisfied only in a region that is external relative to the projection of the crack surface onto the middle plate surface. This property of nonlocality is new in the theory of plates and can serve as the source of a number of new mathematical formulations of boundary-value problems. As for conventional approaches to the description of cracks in elastic (and inelastic) bodies, the literature is extensive.

In the present paper, we shall not discuss these approaches but simply note that they admit the possibility of mutual penetration of the crack sides [4–6]. The properties of boundary-value problems that arise in such cases were analyzed, for example, by Grisvard [7], Ohtsuka [8], Kondrat'ev et al. [9], Oleinik et al. [10], and Nicaise [11]. From the viewpoint of boundary-value problems, it is important to emphasize that, in any case, the presence of cracks (cuts) in a plate leads to the appearance of nonsmooth components of the boundary. The approach considered in [1–3] differs from the traditional ones by the fact that it is characterized by boundary conditions in the form of inequalities at the crack sides. For an oblique crack, the situation becomes even more complicated, because the equilibrium conditions are affected by the opposite crack sides. In reality, this means that additional terms which incorporate the response of opposite sides appear in the equilibrium equations. These terms can be found only after the problem is solved as a whole.

1. Derivation of Impenetrability Conditions. Let the middle surface of a plate occupy the region $\Omega_c \equiv \Omega \setminus \Gamma_c$, where $\Omega \subset R^2$ is the bounded region with a smooth boundary Γ, and Γ_c is the smooth curve without self-intersections which lies in Ω (see Fig. 1). The vertical cross section of the plate is shown in Fig. 2.

The middle plate surface lies in the plane $z = 0$. The coordinate system (x_1, x_2, z) is Cartesian and $x = (x_1, x_2)$.

Let the crack surface Ψ be described by the function $z = \Phi(x)$ $(x \in \Omega_\Psi)$. Here Ω_Ψ is the orthogonal projection of the crack surface [of the graph of the function $z = \Phi(x)$] onto the plane $z = 0$. The normal to the surface $z = \Phi(x)$ $(x \in \Omega_\Psi)$ is denoted by $n(x) = (-\nabla\Phi(x), 1)/\sqrt{1 + |\nabla\Phi(x)|^2}$. The chosen direction of the normal $n(x)$ determines the positive and negative crack sides, denoted by Ψ^+ and Ψ^-, respectively. The curve Γ_c is the intersection of the crack surface Ψ with the plane $z = 0$. For simplicity, it is assumed that $|\nabla\Phi(x)| \neq 0$ $(x \in \Omega_\Psi)$.

The projection Ω_Ψ of the surface Ψ can be naturally presented as a sum of two sets in accordance with the chosen direction of the z axis, namely: $\Omega_\Psi = \Omega_\Psi^+ \cup \Omega_\Psi^-$. We assume that if a part of the surface Ψ

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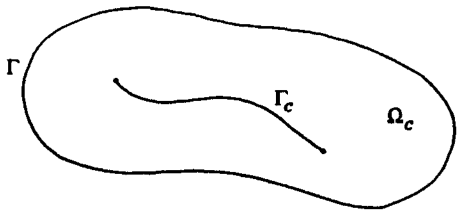

Fig. 1

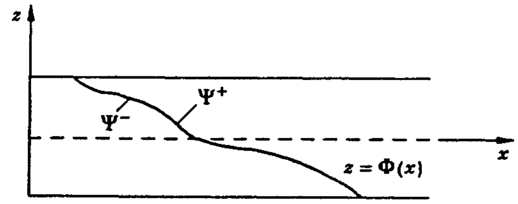

Fig. 2

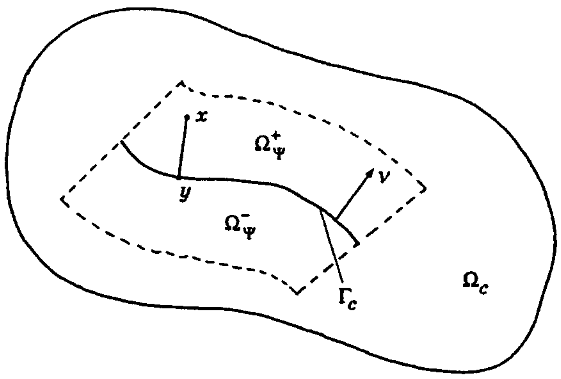

Fig. 3

is projected along the positive direction of the z axis, the corresponding projection of this part is denoted by Ω_Ψ^+, and the projection of a part of Ψ that is projected in the direction opposite to the z axis is denoted by Ω_Ψ^-. In particular, the curve Γ_c belongs to both Ω_Ψ^+ and Ω_Ψ^-.

The direction of the normal $\nu = (\nu_1, \nu_2)$ to the curve Γ_c in the x plane is assumed to be chosen as indicated in Fig. 3.

Let $x \in \Omega_\Psi$. The orthogonal projection of the point x onto the curve Γ_c is denoted by $y = Px$ (see Fig. 3). The sets $\Omega_\Psi^\pm$ are assumed here to be sufficiently small in the sense that the quantity $y = Px$ is defined unambiguously for each $x \in \Omega_\Psi^\pm$.

Recall that in the theory of Kirchhoff–Love plates, horizontal displacements depend linearly on the z coordinate [12]:

$$W(z) = W - z\nabla w, \quad |z| \leqslant 2\varepsilon.$$

Here $W = (w^1, w^2)$ and w are the horizontal and vertical displacements of the points in the middle plate surface, and 2ε is the thickness of the plate. The vector of displacements of the points on the middle surface is denoted by $\chi = (W, w)$ and $\chi = \chi(x)$ ($x \in \Omega_c$). Not leaving the framework of the Kirchhoff–Love hypotheses, we express the displacements of the plate points at the crack sides via $\Psi^\pm$ and derive the condition of mutual impenetrability of the sides.

Let $(x, z) \in \Psi^+$ ($x \in \Omega_\Psi^+$). In accordance with the Kirchhoff–Love formulas, the displacement vector at the point (x, z) is of the form

$$\chi^+(x,z) = (W^+(x) - z\nabla w^+(x), w^+(x)), \quad x \in \Omega_\Psi^+, \quad z = \Phi(x). \tag{1.1}$$

For the points $(x, z) \in \Psi^-$ ($x \in \Omega_\Psi^+$), the displacement vector can be found by the following formula:

$$\chi^-(x,z) = \Big(W^-(y) - z\nabla w^-(y), w^-(y) + |x-y|\frac{\partial w^-(y)}{\partial \nu}\Big), \qquad y = Px. \tag{1.2}$$

Formula (1.2) implies that the horizontal displacements at the point $(x, z) \in \Psi^-$ ($x \in \Omega_\Psi^+$) coincide with those at the point (y, z) ($y = Px$), while the vertical displacements are different from those at the point (y, z) ($y = Px$) by the term $|x - y|\partial w^-(y)/\partial\nu$.

The condition of mutual impenetrability of cracks at the point $(x, z) \in \Psi$ $(x \in \Omega_\Psi^+)$ is as follows:

$$(\chi^+(x,z) - \chi^-(x,z))n(x) \geqslant 0, \qquad x \in \Omega_\Psi^+, \qquad z = \Phi(x). \tag{1.3}$$

According to (1.1) and (1.2), substituting the vectors $\chi^\pm(x,z)$, we obtain

$$(\chi^+(x) - \chi^-(y))n(x) - (\chi_z^+(z) - \chi_z^-(y))n(x) \geqslant 0, \quad x \in \Omega_\Psi^+, \quad z = \Phi(x), \quad y = Px, \tag{1.4}$$

where

$$\chi^\pm(s) = (W^\pm(s), w^\pm(s)); \qquad \chi_z^\pm(s) = \Big(z\nabla w^\pm(s), |s - Ps|\frac{\partial w^\mp(Ps)}{\partial \nu^\pm}\Big).$$

Here we used the following notation: $\partial/\partial\nu^+ \equiv \partial/\partial\nu$ and $\partial/\partial\nu^- \equiv -\partial/\partial\nu$. Note also that, according to the above definitions, $y = Py$ for $y \in \Gamma_c$.

Similarly, if we consider the points $(x, z) \in \Psi^\pm$ $(x \in \Omega_\Psi^-)$, we can derive an impenetrability condition of the form (1.4).

Indeed, let $(x, z) \in \Psi^+$ $(x \in \Omega_\Psi^-)$. Then

$$\chi^+(x,z) = \Big(W^+(y) - z\nabla w^+(y), w^+(y) - |x-y|\frac{\partial w^+(y)}{\partial \nu}\Big), \ x \in \Omega_\Psi^-, \ z = \Phi(x), \ y = Px. \tag{1.5}$$

Similarly, if $(x, z) \in \Psi^-$ $(x \in \Omega_\Psi^-)$, then, in accordance with the Kirchhoff–Love hypotheses,

$$\chi^-(x,z) = (W^-(x) - z\nabla w^-(x), w^-(x)). \tag{1.6}$$

Substituting the values of $\chi^\pm(x,z)$ from (1.5) and (1.6) into the impenetrability condition $(\chi^+(x,z) - \chi^-(x,z))n(x) \geqslant 0$ $[x \in \Omega_\Psi^-$ and $z = \Phi(x)]$, we have

$$(\chi^+(y) - \chi^-(x))n(x) - (\chi_z^+(y) - \chi_z^-(x))n(x) \geqslant 0, \quad x \in \Omega_\Psi^-, \quad z = \Phi(x), \quad y = Px. \tag{1.7}$$

Thus, the condition of mutual impenetrability of the sides of an oblique crack is described by inequalities (1.4) and (1.7), which are of a nonlocal character in the sense that along with the values of the functions at the point x, they contain those at the point $y = Px$ as well, the latter being taken at the opposite side of the crack.

It is important to note the following circumstance. If a crack that is described by the surface $z = \Phi(x)$ is transformed into a vertical crack that corresponds to the cylindrical surface $x \in \Gamma_c$, $-\varepsilon \leqslant z \leqslant \varepsilon$, relations (1.4) and (1.7) are transformed into the known impenetrability condition for vertical cracks [1–3]:

$$[W(x)]\nu(x) \geqslant \varepsilon\Big|\Big[\frac{\partial w(x)}{\partial \nu}\Big]\Big|, \qquad x \in \Gamma_c. \tag{1.8}$$

Here $[V] = V^+ - V^-$ [$V^\pm$ correspond to the V values taken at the positive and negatives sides Γ_c with respect to the direction of the normal (ν_1, ν_2)]. Indeed, in this case the normal $n(x)$ is transformed into the vector $(\nu_1, \nu_2, 0)$, and conditions (1.4) and (1.7) yield, respectively,

$$[W(x)]\nu(x) \geqslant z[\nabla w(x)]\nu(x), \qquad x \in \Gamma_c, \qquad -\varepsilon \leqslant z \leqslant 0; \tag{1.9}$$

$$[W(x)]\nu(x) \geqslant z[\nabla w(x)]\nu(x), \qquad x \in \Gamma_c, \qquad 0 \leqslant z \leqslant \varepsilon. \tag{1.10}$$

Evidently, condition (1.8) is equivalent to (1.9) and (1.10) [2].

Note that if the crack is partially oblique or partially vertical (see Fig. 4), the impenetrability conditions of its sides are of the form (1.4) and (1.10).

Similarly, one can consider other cases of crack obliqueness, for example, the case given in Fig. 5. We omit the corresponding formulas here, because this is easily done using the above considerations.

2. Formulation of the Boundary-Value Problem. Existence of the Solution. We shall consider the boundary-value problem of the equilibrium of a plate containing an oblique crack and prove the existence of the solution. We shall deal with the crack shown in Fig. 2. As is known, the mutual-impenetrability conditions of the crack sides have, in this case, the form of (1.4) and (1.7). Let, as before, $\chi = (W, w)$ be the displacement

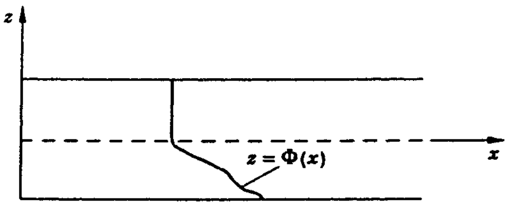

Fig. 4

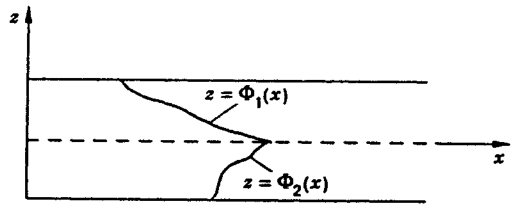

Fig. 5

vector of the points on the middle surface of the plate. We introduce the strain tensor of the middle plate surface

$$\varepsilon_{ij} = \varepsilon_{ij}(W), \quad \varepsilon_{ij}(W) = \frac{1}{2}\Big(\frac{\partial w^i}{\partial x_j} + \frac{\partial w^j}{\partial x_i}\Big) \qquad (i,j = 1,\, 2)$$

and the stress tensor $\sigma_{ij} = \sigma_{ij}(W)$, $i,j = 1,\, 2$, $\sigma_{11} = \varepsilon_{11} + æ\varepsilon_{22}$, $\sigma_{22} = \varepsilon_{22} + æ\varepsilon_{11}$, $\sigma_{12} = (1-æ)\varepsilon_{12}$, $æ = \text{const}$, $0 < æ < 1/2$. The energy functional of the plate is of the form

$$\Pi(\chi) = (1/2)B(W,W) + (1/2)b(w,w) - \langle f, \chi\rangle.$$

Here $B(W,\bar W) = \langle \sigma_{ij}(W), \varepsilon_{ij}(\bar W)\rangle$,

$$b(w,\bar w) = \int\limits_{\Omega_c} (w_{xx}\bar w_{xx} + w_{yy}\bar w_{yy} + æ w_{xx}\bar w_{yy} + æ w_{yy}\bar w_{xx} + 2(1-æ)w_{xy}\bar w_{xy})\,d\,\Omega_c;$$

the brackets $\langle\cdot,\cdot\rangle$ denote integration over Ω_c.

We assume that $f = (f_1, f_2, f_3) \in L^2(\Omega_c)$. Let $H(\Omega_c) = H^{1,0}(\Omega_c)\times H^{1,0}(\Omega_c)\times H^{2,0}(\Omega_c)$, where $H^{s,0}(\Omega_c)$ is the closure of the set of smooth functions, which are equal to zero near Γ, in the norm $H^s(\Omega_c)$. It is easy to see that the set of functions from $H(\Omega_c)$, which satisfy inequalities (1.4) and (1.7), is convex and closed in $H(\Omega_c)$. We denote it by K.

The problem of the equilibrium of a plate with an oblique crack can be posed as a variational one:

$$\inf_{\chi\in K} \Pi(\chi). \tag{2.1}$$

The functional Π is convex and differentiable in the space $H(\Omega_c)$, and, therefore, problem (2.1) is equivalent to the following inequality:

$$\chi \in K\colon \quad \Pi'(\chi)(\bar\chi - \chi) \geqslant 0 \qquad \forall \bar\chi \in K, \tag{2.2}$$

where $\Pi'(\chi)$ is the derivative of the functional Π at the point χ.

By virtue of the inequalities $b(w,w) \geqslant c\|w\|^2_{2,\Omega_c}$ $\forall w \in H^{2,0}(\Omega_c)$ and $B(W,W) \geqslant c\|W\|^2_{1,\Omega_c}$ $\forall W = (w^1, w^2) \in H^{1,0}(\Omega_c)$, the functional Π is coercive in the space $H(\Omega_c)$, i.e., $\Pi(\chi) \to \infty$ $\|\chi\|_{H(\Omega_c)} \to \infty$. Since it is weakly semi-continuous from below, we conclude that the solution of problem (2.1) [or problem (2.2)] exists. The solution will be unique.

There is no difficulty in seeing that in the region $\Omega_c \setminus \Omega_\Psi$, the equations

$$\Delta^2 w = f_3, \qquad -\sigma_{ij,j}(W) = f_i, \qquad i = 1,\, 2, \tag{2.3}$$

are satisfied in terms of distributions. To verify this fact, it suffices to substitute $\chi + \tilde\chi$ into (2.2) as trial functions, where $\tilde\chi = (\tilde W, \tilde w) \in C_0^\infty(\Omega_c \setminus \Omega_\Psi)$ and χ is the solution of problem (2.2). Indeed, this substitution leads to the identity $\Pi'(\chi)(\tilde\chi) = 0$, $\tilde\chi = (\tilde W, \tilde w) \in C_0^\infty(\Omega_c \setminus \Omega_\Psi)$, which means the validity of (2.3) in terms of distributions.

Now let the point x be inner for the set Ω_Ψ^+, i.e., there is a neighborhood U of the point x which belongs to Ω_Ψ^+. In the region Ω_c, we choose a sufficiently smooth function $\tilde\chi = (\tilde W, \tilde w)$ with a carrier in U and such that $(\tilde W^+(x) - z\nabla\tilde w^+(x), \tilde w^+(x))n(x) \geqslant 0$, where $z = \Phi(x)$ and $x \in U$. Then $\chi + \tilde\chi \in K$, where χ is the solution of problem (2.2). We substitute $\chi + \tilde\chi$ into (2.2) as a trial function. We obtain the inequality

$\Pi'(\chi)(\tilde{\chi}) \geqslant 0$, which means that the equilibrium equations (2.3) are not, generally speaking, satisfied at the inner points Ω_Ψ.

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REFERENCES

1. A. M. Khludnev and J. Sokolowski, *Modelling and Control in Solid Mechanics*, Birkhauser Verlag, Basel–Boston–Berlin (1997).
2. A. M. Khludnev, "Contact problem for a cracked shallow shell," *Prikl. Mat. Mekh.*, **59**, No. 2, 318–326 (1995).
3. A. M. Khludnev, "On a contact problem for a plate having a crack," *Contr. Cybern.*, **24**, No. 3, 349–361 (1995).
4. N. F. Morozov, *Mathematical Problems of the Crack Theory* [in Russian], Nauka, Moscow (1984).
5. G. P. Cherepanov, *Fracture Mechanics of Composite Materials* [in Russian], Nauka, Moscow (1983).
6. L. M. Kachanov, *Foundations of Fracture Mechanics* [in Russian], Nauka, Moscow (1974).
7. P. Grisvard, "Singularities in boundary value problems," in: *Research Notes in Applied Mathematics*, Vol. 22, Springer-Verlag, Berlin (1992).
8. K. Ohtsuka, "Mathematical aspects of fracture mechanics," *Lect. Notes Num. Appl. Anal.*, **13**, 39–59 (1994).
9. V. A. Kondrat'ev, I. Kopachek, and O. A. Oleinik, "The behavior of generalized solutions of second-order elliptic equations and of the system of elasticity theory in the neighborhood of a boundary point," in: *Proc. of Petrovskii Seminar* [in Russian], Vol. 8, Izd. Mosk. Univ., Moscow (1982), pp. 135–152.
10. O. A. Oleinik, V. A. Kondrat'ev, and I. Kopachek, "Asymptotic properties of the solutions of a biharmonic equation," *Differ. Uravn.*, **17**, No. 10, 1886–1899 (1981).
11. S. Nicaise, "About the Lamé system in a polygonal or polyhedral domain and the coupled problem between the Lamé system and the plate equation. 1. Regularity of the solutions," *Annali Scuola Norm. Super. Pisa. Ser. IV*, **19**, 327–361 (1992).
12. A. S. Vol'mir, *Nonlinear Dynamics of Plates and Shells* [in Russian], Nauka, Moscow (1972).